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GEOMETRICALLY NON-LINEAR VIBRATION AND MESHLESS DISCRETIZATION OF THE COMPOSITE LAMINATED SHALLOW SHELLS WITH COMPLEX SHAPE

L.V. Kurpa[1]
National Technical University «KhPI»,
Kharkov, Ukraine

ABSTRACT

To study the geometrically non-linear vibrations of the composite laminated shallow shells with complex plan form the approach, based on meshless discretization, is proposed. Non-linear equations of motion for shallow shells based on the first order shear deformation shell theories are considered. The discretization of the motion equations is carried out by method based on expansion of the unknown functions in series for which eigenvectors of the linear vibration obtained by RFM (R-functions method) are employed as basic functions. The factors of these series are functions (generalizing coordinates) depending on time. Due to applying the basic variational principle in mechanics by Ostrogradsky-Hamilton the corresponding system of the ordinary differential equations by Euler is obtained The non-linear ordinary differential equations are derived in terms of amplitudes of the vibration modes. The offered method is expounded for multi-modal approximation of unknown functions. Backbone curves of the spherical shallow shell with complex plan form are obtained using only the first vibration mode by the Bubnov-Galerkin method. The effects of lamination schemes on the behavior are discussed.

INTRODUCTION

Research of the geometrically non-linear vibrations of the laminated plates and open shells of different form is one of important issues of nonlinear dynamics. Due to complexity of the mathematical simulations this problem in general case may be only solved by numerical methods. Many researchers are studying this problem [1,3-5,8,9,11]. Some review of achievements in this field is presented in works [3,8,9]. The main approach which is applied to solve this problem is finite elements method (FEM) combined with method of harmonic balance, Bubnov-Galerkin, multiscales method and another. In studies [6,7] the R-functions method (RFM) has been employed and the new method of the discretization has been proposed. But this approach is effective one for laminated plates and with some accuracy can be applied to higher shallow shells. In the given paper the algorithm of meshless discretization, based on combination of the classical approaches and modern constructive means of the R-functions theory is developed. The considered approach is based on multiple-modes approximation in time of the unknown functions. It allows studying the geometrically non-linear dynamic response of the shallow shells with complex shape and different boundary conditions.

1. FORMULATION OF THE PROBLEM

Let us consider a laminated shallow shell of an arbitrary plan form with radii curvature R_x, R_y which consists of S layers of the constant thickness h_i. The general thickness h is defined as $h = \sum_{i=1}^{S} h_i,$. Assume that shell under consideration has symmetric relatively of the midsurface lamination scheme and its projection in a plane is some domain Ω. Delaminating between the layers is not. Due to shallowness the curvilinear coordinates commonly employed in shells can be directly replaced

[1] Corresponding author. Email Kurpa@kpi.kharkov.ua

by the Cartesian coordinates x and y, and the Lame parameters are: $A=1$, $B=1$. The displacement components at an arbitrary point of the shell are U, V, and W in the x, y and z directions respectively. Investigation we will carry out by first-order shear deformation theory [1,2,9].

According to the first-order shear deformation theory (FSDT) it is assumed that in-plane displacements U and V are linear functions of coordinate z, and that the transverse displacement W is constant through the thickness of the shell.

The normal to the midsurface remains straight after deformation, but not necessarily normal to the middle surface. The non-linear strain-displacement relations of the plates can be written as

$$e_x = \varepsilon_x + z\chi_x,\ e_y = \varepsilon_y + z\chi_y,\ e_{xy} = \varepsilon_{xy} + z\chi_{xy},$$
$$e_z = 0,\ e_{xz} = w,_x - uk_x + \psi_x,\ e_{yz} = w,_y - vk_y + \psi_y$$

where

$$\varepsilon_x = u,_x + \frac{w}{R_x} + \frac{1}{2}w,_x^2,\ \varepsilon_y = v,_y + \frac{w}{R_y} + \frac{1}{2}w,_y^2,\ \varepsilon_{xy} = u,_y + v,_x + w,_x w,_y,$$
$$\varepsilon_{xz} = w,_x + \psi_x,\ \ \varepsilon_{yz} = w,_y + \psi_y,\ \ \chi_x = \psi_x,_x,\ \chi_y = \psi_y,_y,\ \ \chi_{xy} = \psi_x,_y + \psi_y,_x. \quad (1)$$

In these equations the subscripts following comma denote the partial differentiation and u, v and w are the displacements at the midsurface, ψ_x and ψ_y are the rotations about the y- and x-axes respectively. Let us denote as vectors $\{\varepsilon\} = \{\varepsilon_x; \varepsilon_y; \varepsilon_{xy}\}^T$, $\{k\} = \{\psi_x,_x; \psi_y,_y; \psi_x,_y + \psi_y,_x\}^T$, stresses $\{N\} = \{N_x; N_y; N_{xy}\}^T$ and moments resultants $\{M\} = \{M_x; M_y; M_{xy}\}^T$

The constitutive relations of the symmetrically laminated shell can be presented as follows

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} \begin{Bmatrix} \varepsilon \\ k \end{Bmatrix}, \qquad \begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} C_{44} & C_{45} \\ C_{45} & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix}, \quad (2)$$

where

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix},\ D = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}, \quad (3)$$

Constants C_{ij} and D_{ij} (elements of matrices $[C]$ and $[D]$ respectively) are the stiffness coefficients of the shell, which are defined by the following expressions:

$$\left(C_{ij}, D_{ij}\right) = \sum_{m=1}^{S} \int_{h_m}^{h_{m+1}} B_{ij}^{(m)}\left(1, z^2\right)dz, \quad (i, j = 1,2,6), \quad C_{ij} = k_i^2 \sum_{m=1}^{S} \int_{h_m}^{h_{m+1}} B_{ij}^{(m)}dz, \quad (i, j = 4,5)$$

Here $B_{ij}^{(m)}$ are stiffness coefficients of the m-th layer, k_i, $i = \overline{4,5}$ are shear correction factors and h_m is the distance from the midsurface to the upper surface of the m-*th* layer. Usually the value k_i^2, $i = \overline{4,5}$ are taken equal to 5/6. Next, we assume that $k_4 = k_5$, that is. $C_{45} = C_{54}$.

On the other part, $\varepsilon = \varepsilon^{(L)} + \varepsilon^{(NL)}$, where

$$\varepsilon^{(L)} = \left\{u,_x + \frac{w}{R_x}; v,_y + \frac{w}{R_y}; u,_y + v,_x\right\}^T,\ \ \varepsilon^{(Nl)} = \frac{1}{2}\left(w,_x^2; w,_y^2; 2w,_x w,_y\right)^T \quad (4)$$

Vector $N = \{N_x, N_y, N_{xy}\}^T$ can also be written as follows:

$$N = N^{(L)} + N^{(Nl)},\ N^{(L)} = C\varepsilon^{(L)},\ N^{(Nl)} = \cdot C\varepsilon^{(Nl)} \quad (5)$$

If mass density is the same and constant for all layers and layers have the same thickness then the kinetic energy of the shell can be written as

$$T = \frac{\rho h}{2}\iint_{\Omega}\left(\left(u,_{t}^{2} + v,_{t}^{2} + w,_{t}^{2}\right) + \frac{h^2}{12}\left(\psi_{x,t}^{\;2} + \psi_{y,t}^{\;2}\right)\right)d\Omega \tag{6}$$

The strain energy of the shell is given by

$$P = \frac{1}{2}\iint_{\Omega}\left(N_x\varepsilon_x + N_y\varepsilon_y + N_{xy}\varepsilon_{xy} + Q_x\varepsilon_{xz} + Q_x\varepsilon_{xz} + M_x k_x + M_y k_y + M_{xy}k_{xy}\right)d\Omega \tag{7}$$

As shown in works [2, 8, 9] the movement equations may be obtained by Hamilton's principle

$$\int_{t_0}^{t_1}\delta(T - P)dt = 0 \tag{8}$$

Let us write the system of differential equations of the motion in operator form:

$$LU = NL + mU,_{tt}$$

where U, m are vectors $U = \{u; v; w; \psi_x; \psi_y\}^T$, $m = \{m_1, m_1, m_1, m_2, m_2\}^T$,($m_1 = \rho h, \quad m_2 = \dfrac{\rho h^3}{12}$), L is matrix $L = \left[L_{ij}\right]_{\substack{i=\overline{1,5}\\ j=\overline{1,5}}}$ The elements L_{ij} , $i, j = \overline{1,5}$ of the matrix L are linear operators:

$$\begin{aligned}
&L_{11} = C_1\partial^2 - k_1^2 C_{55}, \quad L_{12} = \left(\left(C_{16}, \left(C_{12} + C_{66}\right), C_{26}\right), \partial^2\right) - k_1 k_2 C_{45},\\
&L_{13} = \left(\left(k_1\left(C_{11} + C_{55}\right) + C_{12}k_2, k_1\left(C_{16} + C_{45}\right) + C_{26}k_2\right), \partial\right), \ L_{14} = k_1 C_{55}, \qquad L_{15} = k_1 C_{45},\\
&L_{21} = L_{12}, \ L_{22} = C_3\partial^2 - k_2^2 C_{44}, \quad L_{23} = \left(\left(k_1 C_{16} + k_2\left(C_{26} + C_{45}\right), k_1 C_{12} + k_2\left(C_{22} + C_{44}\right)\right), \partial\right),\\
&L_{24} = k_2 C_{45}, \qquad L_{25} = k_2 C_{44}, \qquad L_{31} = -L_{13}, \qquad L_{32} = -L_{23},\\
&L_{33} = \left(\left(C_{55}, C_{45}, C_{44}\right), \partial^2\right) - k_1^2 - 2k_1 k_2 - k_2^2, \quad L_{34} = \left(\left(C_{55}, C_{45}\right), \partial\right), \quad L_{35} = \left(\left(C_{45}, C_{44}\right), \partial\right),\\
&L_{41} = L_{14}, \quad L_{42} = k_2 C_{44}, \qquad L_{43} = -L_{43}, \qquad L_{44} = D_1\partial^2 - C_{55},\\
&L_{45} = \left(\left(D_{16}, \left(D_{12} + D_{66}\right), D_{26}\right), \partial^2\right) - C_{44}, \quad L_{51} = L_{15}, \ L_{52} = L_{25}, \ L_{53} = -L_{35},\\
&L_{54} = \left(\left(D_{16}, \left(D_{12} + D_{66}\right), D_{26}\right), \partial^2\right) - C_{45}, \quad L_{55} = D_3\partial^2 - C_{44},
\end{aligned}$$

where $\{C_i\}$ and $\{D_i\}$ are i-th rows of the matrixes $[C]$ and $[D]$ relatively, ∂ and ∂^2 are differential operators which are defined as $\partial = \{\partial,_x; \partial,_y\}^T$, $\partial^2 = \{\partial,_{xx}; 2\partial,_{xy}; \partial,_{yy}\}^T$.

The components NL_i , $i = \overline{1,3}$ of the vector $NL = \{NL_1(w), NL_2(w), NL_3(u, v, w), 0, 0\}^T$ are nonlinear operators $NL_1(w) = N_x^{(Nl)},_x + N_{xy}^{(Nl)},_y$, $NL_2(w) = N_{xy}^{(Nl)},_x + N_y^{(Nl)},_x$,

$$Nl_3(u, v, w) = \left(N_x w,_x + N_{xy} w,_y\right),_x + \left(N_{xy} w,_x + N_y w,_y\right),_y - k_1 N_x^{(Nl)} - k_2 N_y^{(Nl)}$$

2. SOLUTION PROCEDURE

Obviously that the first step is reduced to study linear problem in order to find the eigen functions $\{U^{(c)}\} = \{u^{(c)}, v^{(c)}, w^{(c)}, \psi_x^{(c)}, \psi_y^{(c)}\}^T$ satisfying the given boundary conditions and the appropriate natural frequencies of linear vibrations shells. Note that solving linear problem we will not ignore inertia and rotation forces. Solution of linear problems has been widely discussed in works [6,7]. Let us note that in generic case this problem may be solved by RFM [10].

Let us consider in detail the solving non-linear problem. The unknown function are presented as

$$u(x, y, t) = \sum_{k=1}^{n} X_k(t)u_k^{(c)}(x, y), \quad v(x, y, t) = \sum_{k=1}^{n} Y_k(t)v_k^{(c)}(x, y), \quad w(x, y, t) = \sum_{k=1}^{n} Z_k(t)w_k^{(c)}(x, y)$$

$$\psi_x(x, y, t) = \sum_{k=1}^{n} R_{xk}(t)\psi_{xk}^{(c)}(x, y), \quad \psi_y(x, y, t) = \sum_{k=1}^{n} R_{yk}(t)\psi_{yk}^{(c)}(x, y) \tag{9}$$

where $u_k^{(c)}(x,y)$, $v_k^{(c)}(x,y)$, $w_k^{(c)}(x,y)$, $\psi_{xk}^{(c)}(x,y)$, $\psi_{yk}^{(c)}(x,y)$ are *k*-th eigenfunctions of linear vibrations of the shell and $X_k(t), Y_k(t), Z_k(t), R_{kx}(t), R_{ky}(t)$ are unknown functions in time. The following notation is introduced for brevity, $q = \{X_{k,} Y_k, Z_k, R_{xk}, R_{yk}\}^T$, $k = \overline{1,n}$. The generic element of the time-dependent vector $\boldsymbol{q}$ is referred to as q_j. The dimension of $\boldsymbol{q}$ is $\overline{N}$, which is number of freedom used in the mode expansion. In order to obtain the discretized equations we will apply the main variational principle in mechanics (8) by Ostrogradsky-Hamilton. The corresponding system of equations by Euler (or Lagrange eguations [2]) takes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial P}{\partial q_j} = Q_j, \quad j = \overline{1,\overline{N}} \tag{10}$$

where *T* is kinetic energy and *P* is potential energy of the system , Q_j are the generalized forces obtained by differentiation of Rayleigh's dissipation function *F* and the virtual work *W* done by external forces. In the given case we assume that viscous damping is absent (e.g. $F = 0$) , $\frac{\partial T}{\partial q_j} = 0$ So the equations (10) is simplified and taken the following form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial P}{\partial q_j} = \frac{\partial W}{\partial q_j}, \quad j = \overline{1,\overline{N}} \tag{11}$$

The virtual work *W* done by external forces is written as

$$W = \iint_{\Omega} q_z w \, d\Omega$$

Let us put $q_z = \tilde{f}\cos(\omega_e t)$, where ω_e is the excitation frequency, $\tilde{f}$ is force magnitude positive in *z* direction. It should be noted that in nonlinear case we will ignore by inertia and rotation forces. Therefore

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \begin{cases} 0, & j = 1,2,3,\cdots N, \\ 0, & j = N+1, N+2,\cdots 2N, \\ \rho h q_{j,tt} \iint_{\Omega} w_j^2 \, d\Omega, & j = 2N+1, 2N+2,\cdots 3N, \\ 0, & j = 3N+1, 3N+2,\cdots 4N, \\ 0, & j = 4N+1, 4N+2,\cdots 5N \end{cases} \tag{12}$$

$$\frac{dw}{dq_j} = \begin{cases} 0, & j = 1,2,3,\cdots N, \\ 0, & j = N+1,\ N+2,\cdots,2N, \\ \tilde{f}\cos(\omega_e t)\iint_{\Omega} w_j \, d\Omega, & j = 2N+1,\ 2N+2,\cdots,3N, \\ 0, & j = 3N+1,\ 3N+2,\cdots,4N, \\ 0, & j = 4N+1,\ 4N+2,\cdots,5N \end{cases} \tag{13}$$

$$\frac{\partial P}{\partial q_j} = \sum_{i=1}^{\overline{N}} f_{ji} q_i + \sum_{i=1}^{\overline{N}} \sum_{k=1}^{\overline{N}} f_{jik} q_i q_k + \sum_{i=1}^{\overline{N}} \sum_{k=1}^{\overline{N}} \sum_{l=1}^{\overline{N}} f_{jikl} q_i q_k q_l \tag{14}$$

Substituting the expressions (12-14) into equations (11) we can see that equation, corresponding $j = \overline{1,\ 2N}$, and $j = \overline{3N+1,\ 5N}$ are homogeneous algebraic equations in variables $q_j(t)$. Therefore it is possible to find the dependence between vectors $X(t), Y(t), R_x(t), R_y(t)$ and $Z(t)$ in formulas (9). As result we obtain the system of *n* nonlinear differential equations in variables $Z_j(t)$ of the following type:

$$Z''_j(t)+\alpha_j Z_j(t)+\sum_{i=1}^{n}\sum_{k=1}^{n}\beta_{jik} Z_i(t)Z_k(t)+\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\gamma_{jikl} Z_i(t)Z_k(t)Z_l(t)=\gamma_j \tilde{f}\cos\omega_e t\,,$$

$$j=\overline{1,n}$$

The solving obtained system of ordinary differential equations can be performed using various approximate methods, such as the harmonic balance method (HBM), multiscale method, method of the Runge-Kutta, Bubnov-Galerkin and others.

The implementation of the proposed method will be carry out in framework POLE-RL system and MATLAB.

3. NUMERICAL RESULTS

The foregoing method was tested for shallow shells supported on rectangular plan form and

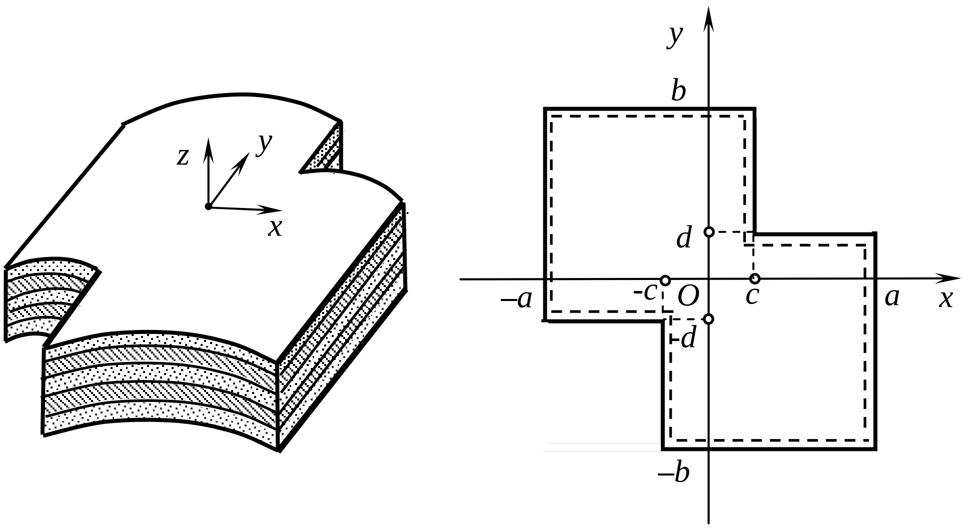

Fig. 1. The shallow shell with complex shape

obtained results have been in good agreement with available ones. Below we present the results for simply supported shells with plan form shown in Fig. 1.

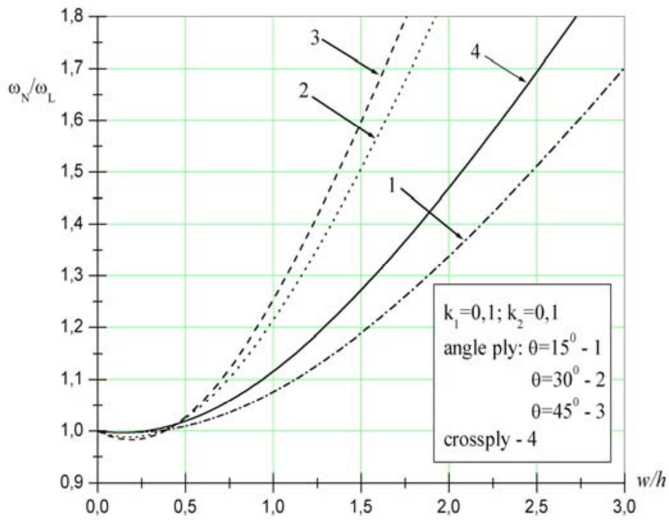

Fig.2. Backbone curves for spherical shell

It is assumed that the shell has five layers which are symmetrical relatively of middle surface. It is made of a material with the following mechanical characteristics: $E = 25E_2,\ G_{12} = G_{13} = 0.5E_2,\ G_{23} = 0.2E_2,\ \nu_{12} = 0.25$. The shear factors are taken to be $k_4^2 = k_5^2 = 5/6$. The geometric parameters of the shell are taken as follows $b/a = 1,\quad c/(2a) = d/(2a) = 0.25,\quad 2a/R_x = 2a/R_y = 0.1,\quad h/(2a) = 0.1$. The lamina scheme is: angle-ply $\left(\theta^0/-\theta^0/\theta^0/-\theta^0/\theta^0\right)$ and cross-ply $\left(0^0/90^0/0^0/90^0/0^0\right)$. For solving the procedure by Bubnov-Galerkin has been applied using only one-mode approximation.

In Fig. 2 the backbone curves for angle-ply $\vartheta = 15^0, 30^0, 45^0$ and cross-ply spherical shells are presented. From analysis backbone curves it follows that the behavior of the curves has soft type for angle-ply spherical shells if $\theta = 30^0$ and $\theta = 45^0$, $0 \prec w_{\max}/h \prec 0.4$. For cross-ply shells and plate the corresponding backbone curves have a hard type.

CONCLUSIONS

A numerically-analytic method is proposed to solve nonlinear vibration problems for symmetric laminated shallow shells with complex plan form. The method is worked out in frame of the refined shell theory of the first order taking shear deformation into account, and geometric nonlinear theory by von Karman-type. The created method is based on the R-functions theory and variational methods. Using multi-model approximation the initial problem has been reduced to system of the nonlinear ordinary differential equation, which may be solved by numerical approach. The method is illustrated on example of the five-layer simply supported spherical and cylindrical shallow shells which is rested on the plan of the complicated form. The layers of shell under consideration have the different lamina schemes. Effect of curvatures is studied for backbone curves.

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